Solution 8

1. Solution:

$$Cov(X+Y, X-Y) = cov(X,X) + cov(X,-Y) + Cov(Y,X) + Cov(Y,-Y)$$
$$= var(X) - cov(X,Y) + Cov(Y,X) - var(Y)$$
$$= var(X) - var(Y)$$
$$= 0 \quad \because X \text{ and } Y \text{ are identically distributed.}$$

$$Cov(X,Y|Z)$$
$$= E\{XY - E[X|Z]Y - XE[Y|Z] + E[X|Z]E[Y|Z] \,|\, Z\}$$
$$= E[XY|Z] - E[X|Z]E[Y|Z] - E[X|Z]E[Y|Z] + E[X|Z]\cdot E[Y|Z] \quad \because E[\cdot] \text{ is a constant}$$
$$= E[XY|Z] - E[X|Z]E[Y|Z]$$

Take the expectation on both sides of result in part i:

$$E[Cov(X,Y|Z)] = E\{E[XY|Z] - E[X|Z]E[Y|Z]\}$$
$$E[Cov(X,Y|Z)] = E[XY] - E\{E[X|Z]E[Y|Z]\} \quad \text{—} ①$$

consider $cov(E[X|Z], E[Y|Z])$.

$$cov(E[X|Z], E[Y|Z]) = E\{E[X|Z]E[Y|Z]\} - E\{E[X|Z]\}\cdot E\{E[Y|Z]\}$$
$$cov(E[X|Z], E[Y|Z]) = E\{E[X|Z]E[Y|Z]\} - E[X]E[Y] \quad ②$$

Combine ① and ②:

$$E[cov(X,Y|Z)] + cov(E[X|Z], E[Y|Z])$$
$$= E[XY] - E(X)E(Y)$$
$$= Cov(X,Y) \qquad \text{Q.E.D.}$$

iii) $cov(X,Y|Z) = Var(X|Z)$ when $Y = X$.

The LHS of result in part ii:
$$cov(X,Y) = var(X)$$

The RHS: $E[cov(X,X|Z)] + cov(E(X|Z), E(\ |Z))$
$$= E[var(X|Z)] + var(E[X|Z])$$
$$\Rightarrow \quad var(X) = E[var(X|Z)] + var(E[X|Z])$$

2. Solution:

$$r_y(0) = \frac{1}{1-b^2}$$

$$r_y(1) = \frac{-a}{1+b}\cdot\frac{1}{1-b^2}$$

$$r_y(2) = \left(-b + \frac{a^2}{1+b}\right)\cdot\frac{1}{1-b^2}$$